THE FUNCTIONAL EQUATION $g(x^2) = 2\alpha x + [g(x)]^2$.

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1. **Introduction.** The functional equation considered in this paper arose out of an extension of a problem in arrangements which occurs in the theory of linear algebras. In an algebra which is neither associative nor commutative, n factors may be associated in a number of different ways; thus for four factors a_1, a_2, a_3, a_4 we have five different associations in which the subscripts all occur in their natural order, namely

$$a_1\,(a_2\cdot a_3\,a_4),\qquad (a_1\,a_2)\,(a_3\,a_4),\qquad (a_1\cdot a_2\,a_3)\,a_4,$$
$$a_1\,(a_2\,a_3\cdot a_4),\qquad\qquad (a_1\,a_2\cdot a_3)\,a_4.$$

The first problem then is to determine the number N_n of such *types* of association for n factors.

It is easily seen* that we can count the number of different types by taking first those in which the left-hand factor consists of one element and the right-hand one of n—1, then those in which the first has two elements and the second n—2, and so on. Hence

$$N_n = N_1\,N_{n-1} + N_2\,N_{n-2} + \cdots + N_{n-1}\,N_1.$$

If we set

$$f(x) = N_1 x + N_2 x^2 + N_3 x^3 + \cdots,$$

we have, since $N_1 = 1$,

$$f^2(x) = f(x) - x,$$

and therefore, since $f(x)$ vanishes for $x = 0$,

$$f(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_1^\infty \frac{(2n-2)!}{(n-1)!\,n!}\,x^n,$$

so that

$$N_n = \frac{(2n-2)!}{(n-1)!\,n!}.$$

* This solution (for an equivalent problem) is also given by P. Quarra, Torino Atti vol. 53, (1918) pp. 1044-1047. See also P. Franklin, Question 2681, Amer. Math. Monthly vol. 25 (1918), p. 118 and solution by C. F. Gummer, ibid., vol. 26 (1919) pp. 127-128.

If we assume that multiplication is commutative, the problem of counting types is much more difficult as is seen from the fact that for $n = 4$

$$a_1\,(a_2 \cdot a_3\, a_4), \qquad (a_1\, a_2)\,(a_3\, a_4)$$

are the only types since $a_1 \cdot a_2\, a_3 = a_2\, a_3 \cdot a_1$ are now of the same type. If n is odd, the method of counting used above is still valid and leads to

$$(1) \qquad 2\,N_n = N_1\,N_{n-1} + N_2\,N_{n-1} + \cdots + N_{n-1}\,N_1,$$

since the types corresponding to the term $N_{n-r}\,N_r$ are the same as those belonging to the term $N_r\,N_{n-r}$. If, however, n is even, say $n = 2k$, the number of types in which the number of elements in the left and right factors is $k = n/2$ is obviously not $N_k^2/2$ but $N_k\,(N_k + 1)/2$, so that we have in place of (1)

$$(2) \qquad 2\,N_n = N_1\,N_{n-1} + N_2\,N_{n-2} + \cdots + N_{n-1}\,N_1 + N_{\frac{n}{2}}.$$

If, on the analogy of the solution in the previous problem, we set

$$g(x) = 1 - \sum_1^\infty N_n x^n,$$

we readily derive from (1) and (2) that $g(x)$ satisfies the functional equation

$$g(x^2) = 2x + g^2(x).$$

This equation may be replaced by another of somewhat simpler form by setting

$$(3) \qquad h(x) = g(x)/x^{\frac{1}{2}},$$

which gives

$$h(x^2) = 2 + h^2(x),$$

from which it is obvious that $\varphi_z = h(x^{2^z})$ is a solution of the difference equation

$$\varphi_{z+1} = 2 + \varphi_z^2.$$

This suggests the consideration of the difference equation

$$(4) \qquad \psi_{z+1} = a\psi_z^2 + b\psi_z + c$$

where a, b and c are constants and $a \neq 0$. On setting

$$\varphi_z = a\psi_z + \frac{b}{2}, \qquad \alpha = \frac{4ac + 2b - b^2}{8},$$

equation (4) becomes

$$\varphi_{z+1} = 2\alpha + \varphi_z^2,$$

which, as above, leads to the consideration of the functional equation

$$g(x^2) = 2\alpha x + g^2(x) \tag{5}$$

with which we shall be mainly concerned here. The associated function defined by (3) then satisfies the equation

$$h(x^2) = 2\alpha + h^2(x). \tag{6}$$

When it is desired to indicate expressly the dependence of these functions on α, we shall write $g(x, \alpha)$ and $h(x, \alpha)$ in place of $g(x)$ and $h(x)$.

Equations (5) and (6) belong to a class of functional equations considered by Poincaré and Picard. The reader is referred to a series of papers by Fatou* where full references are given.

2. **Solutions which are regular at the origin.** If $g(x)$ is regular at the origin, the value a_0 of $g(0)$ satisfies the equation $a_0^2 = a_0$, so that it is either 0 or 1. If $a_0 = 0$, then also $\alpha = 0$, and it is readily seen that $g(x) = x^m$, m arbitrary. Excluding this exceptional case, we may therefore set

$$g(x) = 1 - a_1 x - a_2 x^2 - \cdots = 1 - \sum_1^\infty a_n x^n \tag{7}$$

in (5) and, comparing coefficients,

$$\begin{aligned}
a_1 &= \alpha \\
2a_2 &= a_1 a_1 + a_1 \\
2a_3 &= a_1 a_2 + a_2 a_1 \\
2a_4 &= a_1 a_3 + a_2 a_2 + a_3 a_1 + a_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
2a_{2k} &= a_1 a_{2k-1} + a_2 a_{2k-2} + \cdots + a_{2k-1} a_1 + a_k \\
2a_{2k+1} &= a_1 a_{2k} \quad + a_2 a_{2k-1} + \cdots + a_{2k} a_1,
\end{aligned} \tag{8}$$

* P. Fatou. Sur les équations fonctionnelles, Bull. Soc. Math. de France, vol. 47, pp. 161–271, vol. 48, pp. 33–94, 208–384.

which may be written

$$(8') \qquad a_1 = \alpha, \qquad 2\,a_m = \sum_{r=1}^{m-1} a_r\, a_{m-r} + a_{m/2}, \qquad (m = 2, 3, \ldots)$$

if we agree to reckon $a_{m/2}$ as zero when $m/2$ is not an integer.

There are two exceptional values of α in which the solution (7) is trivial; firstly $\alpha = 0$, which gives $g(x) \equiv 1$, and secondly $\alpha = -1$, in which case all the a's vanish except the first and

$$g(x) = 1 + x, \qquad (\alpha = -1).$$

The case $\alpha = 0$ is excluded from further consideration unless specially mentioned and, of course, in the case $\alpha = -1$ the discussion in the remaining sections is trivial.

3. **The convergence of the series for $g(x)$.** If the term $a_{m/2}$ in (8′) is suppressed, equation (5) becomes

$$(9) \qquad g^2(x) = -2\,\alpha x + 1,$$

or

$$g(x) = \sqrt{1 - 2\,\alpha x} = 1 - \alpha x - \tfrac{1}{2}\,\alpha^2 x^2 - \cdots$$
$$= 1 - \sum_1^{\infty} b_n\, x^n;$$

this series converges for $|x| < 1/2\,|\alpha|$. If we set $\beta_n = |b_n|/2$, then, since each b is the product of a power of α and a positive numerical coefficient, we see readily from (9) that

$$(10) \qquad \begin{aligned} \beta_1 &= |\alpha|/2 \\ \beta_2 &= \beta_1\, \beta_1 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \beta_m &= \beta_1\, \beta_{m-1} + \beta_2\, \beta_{m-2} + \cdots + \beta_{m-1}\, \beta_1 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

and conversely (10) leads to a convergent series.

If we set $|a_n| = \alpha_n$, then from (8)

$$(11) \qquad \alpha_1 = |\alpha|, \quad 2\,\alpha_m \leqq \alpha_1\, \alpha_{m-1} + \alpha_2\, \alpha_{m-2} + \cdots + \alpha_{m-1}\, \alpha_1 + \alpha_{m/2},$$

and, if γ_n is the sequence of positive numbers defined by

$$\gamma_1 = |\alpha|, \quad 2\gamma_m = \gamma_1\gamma_{m-1} + \gamma_2\gamma_{m-2} + \cdots + \gamma_{m-2}\gamma_1 + \gamma_{m/2}$$

then

$$\alpha_r \leqq \gamma_r, \quad (r = 1, 2, \ldots). \tag{12}$$

For, assuming that this inequality holds for $r < m$, as is certainly the case for $m = 2$, then from (11)

$$2\alpha_m \leqq \sum_1^{m-1} \alpha_r \alpha_{m-r} + \alpha_{m/2} \leqq \sum_1^{m-1} \gamma_r \gamma_{m-r} + \gamma_{m/2} = 2\gamma_m,$$

so that (12) follows by induction. The series (7) for $g(x, \alpha)$ therefore converges if $\sum \gamma_n x^n \equiv g(x, |\alpha|)$ converges.

Let δ be a positive quantity satisfying the conditions

$$\delta > |\alpha|, \qquad \delta > 1,$$

and let δ_m be the sequence of increasing positive numbers defined by

$$\delta_1 = \delta, \qquad \delta_m = \sum_1^{m-1} \delta_r \delta_{m-r}, \qquad (m = 2, 3, \ldots),$$

then

$$\gamma_m \leqq \delta_m.$$

For this inequality is true for $m = 1$ and, assuming that it is true for $r < m$, we have

$$2\gamma_m = \sum_1^{m-1} \gamma_r \gamma_{m-r} + \gamma_{m/2} \leqq \sum_1^{m-1} \delta_r \delta_{m-r} + \delta_{m/2} = \delta_m + \delta_{m/2} < 2\delta_m,$$

since the sequence of δ's continually increases. Now we have already seen in (10) that $\sum \delta_r x^r$ converges for $|x| < 1/4\delta$; hence $\sum \gamma_n x^n$ converges, and therefore also $\sum \alpha_n x^n$, that is to say, the series (7) converges absolutely for

$$|x| < 1/4|\alpha|, \qquad |x| < 1/4.$$

We have therefore proved that there always exists a unique solution of (5) which is regular at the origin.

When α is real and positive, an upper limit for the radius of convergence may be found as follows. The first three terms in the series for $g(x)$ are

$$1-\alpha x-\frac{\alpha(\alpha+1)}{2}x^2,$$

the remaining terms all having negative coefficients when $\alpha>0$; hence $g(\alpha^{-1})$ is negative, if the series converges for that value of x. Now $g(0)=1>0$, so that, if the radius of convergence, R, is greater than $1/\alpha$, there is some value ζ of x for which $g(\zeta)=0$; and, if we put $x=\zeta^{\frac{1}{2}}$ in (5), we have

$$0=g(\zeta)=2\alpha\zeta^{\frac{1}{2}}+g^2(\zeta^{\frac{1}{2}}),$$

so that $g(\zeta^{\frac{1}{2}})$ is imaginary. Since this is impossible so long as both ζ and $\zeta^{\frac{1}{2}}$ are inside the circle of convergence, R is certainly less than the smaller of $1/\alpha$ and $1/\alpha^{\frac{1}{2}}$. In particular, if $\alpha>1$, then $R<1$. Closer limits are of course obtained by taking more terms of the series. For instance, if $\alpha=1$ and $x=2/3$, the sum of the first three terms of the series is negative and therefore the same argument as before shows that $R<2/3$. Similarly if $\alpha>(\sqrt{13}-3)/2=0.3027\ldots$, the radius R is less than 1.

4. **The singularities of $g(x)$.** If we write (5) in the form

$$g(x)=2\alpha x^{\frac{1}{2}}+g^2\left(x^{\frac{1}{2}}\right), \tag{13}$$

we see immediately that, if $x=\zeta$ is a singularity, so is also $x=\zeta^{\frac{1}{2}}$; and similarly all the points $\zeta^{2^{-n}}$ $(n=1,2,\ldots)$ are singularities. If $|\zeta|=1$, all these points lie on the circle* C_1, while, if $|\zeta|\neq 1$, the points approach more and more closely to this circle as n increases and at the same time become more and more numerous, since there are 2^n determinations of $\zeta^{2^{-n}}$, and moreover in such a way that every point of C_1 is a limit point of the set of singularities. The circle C_1 therefore forms a natural boundary across which $g(x)$ cannot be continued analytically. Hence *the radius of convergence of the series* (7) *is never greater than* 1 *unless it is infinite.*†

In exactly the same way, using (5) in place of (13), it follows that, if ζ is a singularity, so is also ζ^2 unless $g^2(x)$ is regular at $x=\zeta$ which is then a branch point of order 2 at which $g(x)=0$. If ζ is a singularity for which

* The circle with center at the origin and radius $|x|=r$ will be denoted by C_a or C_r.
† It is shown below that the radius is only infinite when $\alpha=-1$ and $g(x)=1+x$.

$|\zeta|$ is a minimum and we assume $|\zeta| < 1$, then $|\zeta| > 0$ and, since $|\zeta^2| < |\zeta|$, it follows that $g^2(x)$ is regular at $x = \zeta$, which is therefore a branch point of order 2 at which $g(x) = 0$. We may therefore set

$$g(x) = (x - \zeta)^{\frac{1}{2}} g_1(x - \zeta),$$

where $g_1(x)$ is regular at $x = \zeta$. Moreover, if $|\zeta|$ is not a minimum but is still less than 1, then, since $|\zeta^n| \to 0$ as $n \to \infty$ while $g(x)$ is regular at $x = 0$, $g(x)$ must be regular at ζ^n for some value of n. It then follows from (5) that $g(\zeta)$ is finite and has a finite number of determinations; such singularities are therefore algebraic. We shall now show that, if there is a singularity inside C_1, there is a *unique* singularity for which $|\zeta|$ is a minimum.

We have already seen in (6) that $h(x) = g(x)/x^{\frac{1}{2}}$ satisfies the equation

$$h(x^2) = 2\alpha + h^2(x). \tag{14}$$

If x_1 and x_2 are two different zeros of $g(x)$ which lie inside C_1, then

$$h(x_1^{2^n}) = h(x_2^{2^n}), \tag{15}$$

as each is the same polynomial in 2α, e. g., if in (14) $h(x) = 0$, then

$$h(x^2) = 2\alpha,$$

$$h(x^4) = 2\alpha + h^2(x^2) = 2\alpha + 4\alpha^2,$$

and so on. Now $f(x) = 1/h^2(x) = x/g^2(x)$ is regular at $x = 0$, and

$$f'(x) = \frac{g(x) - 2xg'(x)}{g^3(x)},$$

which has the value 1 at $x = 0$. But in (15), $|x_1^{2^n}|$ and $|x_2^{2^n}|$ can be made as small as we please by making n sufficiently large; hence there are an infinity of distinct pairs of points x' and x'' in any neighbourhood of the origin for which

$$\frac{f(x') - f(x'')}{x' - x''} = 0.$$

This is impossible since $f'(0) = 1$; hence $g(x)$ vanishes for at most one value of x within C_1. We have already shown that $g(x)$, and therefore also $h(x)$,

vanishes at the branch point nearest the origin and hence it follows that there is not more than one such point.

We have therefore shown that $g(x)$ *has no singularities within the circle* C_1 *except possibly branch points of finite order; and, if it has any singularity within* C_1, *there is a unique singularity* ζ *for which* $|\zeta|$ *is a minimum; this point is a branch point of order 2 at which* $g(x)=0$ *and it is the only zero within* C_1. *Every point* $\zeta^{1/2^n}$ *is also a singular point and every singular point within* C_1 *is of this form.*

Exactly the same argument as above may be used to show that $h(x)$ never takes on the same value twice within C_1.

We have seen above that C_1 in general forms a natural boundary for $g(x)$. There exist, however, solutions which have a simple pole at $x=\infty$ but otherwise behave in the region exterior to C_1 in much the same way as $g(x)$ does in the interior region, or, more explicitly, if $\zeta_1, \zeta_2, \ldots$ are the singularities of $g(x)$ within C_1, then

$$\bar{g}(x) = xg\left(\frac{1}{x}\right)$$

is a solution of (5) whose only singularities outside C_1 are $\zeta_1^{-1}, \zeta_2^{-1}, \ldots$ and a simple pole at $x=\infty$. For, if $|x| > |\zeta^{-1}|$, ζ being as before the singularity of $g(x)$ whose modulus is least, then

$$\begin{aligned}\bar{g}(x^2) &= x^2 g\left(\frac{1}{x^2}\right) = x^2\left[\frac{2\alpha}{x} + g^2\left(\frac{1}{x}\right)\right] \\ &= 2\alpha x + \left[xg\left(\frac{1}{x}\right)\right]^2 = 2\alpha x + \bar{g}^2(x).\end{aligned}$$

In particular, when $g(x) = 1+x$, we have $\bar{g}(x) \equiv g(x)$.

It is readily seen that no entire function except $1+x$ can be a solution of (5). For differentiating the terms of this equation we have

$$\begin{aligned}g(x^2) &= 2\alpha x + [g(x)]^2 \\ 2xg'(x^2) &= 2\alpha + 2gg', \\ 4x^2 g''(x^2) + 2g'(x^2) &= 2gg'' + 2g'^2 \\ &\cdots\cdots\cdots\cdots\end{aligned}$$

$$\begin{aligned}2^n x^n g^{(n)}(x^2) + \frac{n(n-1)}{2} 2^{n-1} x^{n-2} g^{(n-1)}(x^2) + \cdots \\ = 2g(x)g^{(n)}(x) + 2ng'(x)g^{(n-1)}(x) + \cdots\end{aligned}$$

Hence the value of $g^{(n)}(1)$ is determined uniquely in terms of $g(1)$ unless $g(1) = 2^{n-1}$, i. e., $2\alpha = 2^{n-1} - 2^{2n-2}$ for any positive integral value of n. For these values of α, we have, on putting $x = -1$ in (5),

$$g(1) = -2^{n-1} + 2^{2n-2} + [g(-1)]^2$$

or

$$[g(-1)]^2 = 2^{n-1} + 2^{n-1} - 2^{2n-2} = -(2^{2n-2} - 2^n) < 0$$

unless $n = 1, 2$. Since the series for $g(x)$ has real values when α and x are real, $g(-1)$ cannot have a negative square so that we need only consider the cases $n = 1, 2$. For $n = 2$, we have $\alpha = -1$, and, as we have already seen, $g(x) = 1 + x$ which is an entire function; for $n = 1$, $\alpha = 0$ and $g(x) = 1$, again an entire function; we may therefore exclude these two trivial cases.

Now $\bar{g}(x)$ satisfies the same functional equation as $g(x)$ and $\bar{g}(1) = g(1)$, and therefore the derivatives of $\bar{g}(x)$ have the same values at $x = 1$ as those of $g(x)$. But, if $g(x)$ is an entire function, $\bar{g}(x)$ is regular at $x = 1$; it follows therefore that the series defining $g(x)$ and $\bar{g}(x)$ are identical. This is impossible as $g(x)$ is regular at $x = 0$ while $\bar{g}(x)$ has a singularity there except for the cases $\alpha = 0,1$ considered above. We have therefore shown that $g(x)$ *is not an entire function except when* $\alpha = 0$, $g(x) = 1$, *and* $\alpha = -1$, $g(x) = 1 + x$.

5. **The radius of convergence.** The radius of convergence of (7) can be calculated by means of (14) under certain restrictions and, when α is real, also the value of x which corresponds to a given real value of $g(x)$.

By successive applications of (14) we have

$$(16)\qquad \begin{aligned} h(x^2) &= 2\alpha + h^2(x) \\ h(x^4) &= 2\alpha + (2\alpha + h^2(x))^2 \\ &\cdots\cdots\cdots\cdots \\ h(x^{2^n}) &= 2\alpha + (2\alpha + \cdots + (2\alpha + h^2(x))^2 \cdots)^2, \end{aligned}$$

where in $h(x^{2^n}) = g(x^{2^n})/x^{2^{n-1}}$ the term 2α occurs n times. If we set

$$(17)\qquad k_n(x) = |h(x^{2^n})|^{\frac{1}{2^{n-1}}} = \frac{|g(x^{2^n})|^{\frac{1}{2^{n-1}}}}{|x|},$$

then, since, if $|x| < 1$, $x^{2^n} \to 0$ as $n \to \infty$ and $g(0) = 1$, it follows that

$$(17') \qquad \lim_{n \to \infty} k_n(x) = \frac{1}{|x|}.$$

If we replace $h(x)$ by λ in (16) and write*

$$(18) \qquad \begin{aligned} h_0 &= \lambda, & k_0 &= |\lambda^2|, \\ h_1 &= 2\alpha + \lambda^2, & k_1 &= |2\alpha + \lambda^2|, \\ &\cdots\cdots & &\cdots\cdots \\ h_n &= 2\alpha + h_{n-1}^2, & k_n &= |h_n|^{\frac{1}{2^{n-1}}}, \end{aligned}$$

then k_n approaches a definite limit as $n \to \infty$ provided λ is admissable as a value of $h(x)$ for $|x| < 1$. The convergence of k_n to its limit is, as a rule, rapid for real values of α which are not too small, and this furnishes a practical method of calculating the zero $x = \zeta$, which corresponds to $\lambda = 0$, provided we are able otherwise to determine that it lies inside C_1.

The values of ζ calculated in this manner for certain values af α are

α	ζ	α	ζ
0.25	0.9292	5.00	0.095 33
0.50	0.6654	10.00	0.048 80
1.00	0.4027	50.00	0.009 95
2.00	0.2230		

The convergence of k_n to its limit can also be determined independently under certain conditions. We shall suppose in the first case that for some value of r

$$(19) \qquad |h_r| \gtreqq 1 + |2\alpha|^{\frac{1}{2}}$$

so that

$$|h_r|^2 \gtreqq 1 + 2|2\alpha|^{\frac{1}{2}} + |2\alpha| > |2\alpha|;$$

then

$$|h_{r+1}| = |2\alpha + h_r^2| > |h_r|^2 - |2\alpha| \gtreqq 1 + 2|2\alpha|^{\frac{1}{2}},$$

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* When it is necessary to indicate the values of λ and 2α explicitly, we shall write $h_n(\lambda, 2\alpha)$ in place of h_n.

whence it follows easily by induction that

$$(20) \qquad |h_{r+n}| \overline{\overline{>}}\, 1 + 2^n\,|2\alpha|^{\frac{1}{2}};$$

and therefore if n is sufficiently large,* $|h_n| > |2\alpha|$.
Now

$$\frac{k_n}{k_{n-1}} = \left|\frac{h_n}{h_{n-1}^2}\right|^{\frac{1}{2^{n-1}}} = \left|1 + \frac{2\alpha}{h_{n-1}^2}\right|^{\frac{1}{2^{n-1}}} \leqq \left(1 + \frac{|2\alpha|}{|h_{n-1}^2|}\right)^{\frac{1}{2^{n-1}}}$$

$$\leqq (1 + |2\alpha|)^{\frac{1}{2^{n-1}}} < 1 + \frac{|2\alpha|}{2^{n-1}}$$

for n sufficiently large. The infinite product $\prod k_n/k_{n-1}$ therefore converges and so k_n approaches a definite, finite limit.

If $|2\alpha| \overline{\overline{>}}\, 2.25$, it is easily shown that (19) is satisfied for $\lambda = 0$ and $r = 2$.

The convergence follows in the same way if it is known that $|h_n| \overline{\overline{>}}\, \varepsilon > 0$ for all values of $n > r$, ε being independent of n.

If $\lambda = 0$ is a possible value of λ and z, $|z| < 1$, the corresponding value of x, then as before

$$h(z^2) = 2\alpha,$$

$$h(z^4) = 2\alpha + (2\alpha)^2$$

$$\cdots\cdots\cdots$$

$$h(z^{2^n}) = 2\alpha + h^2(z^{2^{n-1}}).$$

Hence $h(z^{2^n})$ is a polynomial in α of degree 2^{n-1} whose coefficients are positive integers which do not depend on z or α; this polynomial we shall denote by $p_n(2\alpha)$. If $\mu = 2\alpha$ is a root of $p_n(\mu) = 0$, then $h(z^{2^n}) = 0$, so that z^{2^n} is a root of $g(x)$. Hence $(z^{2^n})^{2^n}$ is also a root, and so on. This, however, is impossible if $|z| < 1$, as the origin would then be an essential singularity of $g(x)$; we can therefore conclude that, *if 2α is a root of any of the polynomials $p_n(\mu)$, $g(x)$ vanishes at no point within C_1 which is therefore the circle of convergence of the series* (7), except in the two trivial cases in which this series represents an entire function.

6. **The polynomials $h_n(\lambda, \mu)$ and $p_n(\mu)$.** The polynomial $h_n(\lambda, \mu)$ is the polynomial defined in (18) with μ in place of 2α, i.e.,

$$(21) \qquad h_0(\lambda, \mu) = \lambda, \qquad h_n(\lambda, \mu) = \mu + h_{n-1}^2(\lambda, \mu),$$

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* If $|h_n| > 1 + |2\alpha|^{\frac{1}{2}}$ for every n greater than a certain value, then evidently $\lim k_n \leqq 1$.

while $p_n(\mu) = h_n(0, \mu)$ so that

$$(21') \qquad p_1(\mu) = \mu, \qquad p_n(\mu) = \mu + p_{n-1}^2(\mu).$$

The following properties follow readily from these definitions.

In the first place we have

$$(22) \qquad h_n(\lambda, \mu) = h_{n-r}(h_r(\lambda, \mu), \mu) = p_{n-r}(\mu) + q_{nr}(\lambda, \mu)\, h_r^2(\lambda, \mu),$$

where $q_{nr}(\lambda, \mu)$ is a polynomial in λ, μ with positive integral coefficients; hence in particular

$$(22') \qquad p_n(\mu) = p_{n-r}(\mu) + q_{nr}(0, \mu)\, p_r^2(\mu).$$

From these equations we deduce immediately that

(i) *$h_n(\lambda, \mu)$ and $h_r(\lambda, \mu)$ have no common factor; and if s is the H. C. F. of n and r, $p_s(\mu)$ is the H. C. F. of $p_n(\mu)$ and $p_r(\mu)$.*

To prove the first part of this lemma we need only observe that from (22) every common factor of h_n and h_r is a factor of $p_{n-r}(\mu)$ and therefore does not contain λ. This is, however, impossible since the coefficient of the highest power of λ in h_r is unity.

To prove the second part we observe from (22′) that every common factor of p_n and p_r is a factor of p_{n-r}; and hence, by a repetition of this argument, it is a factor of p_s. It only remains, therefore, to prove that p_s is a factor of p_{ks}, k being any integer. Putting $n = 2s, 3s, \ldots, r = s, 2s, \ldots$ in (22′), we have

$$p_{2s} = p_s + q_{2s,s}\, p_s^2, \qquad p_{3s} = p_{2s} + q_{3s,s}\, p_s^2,$$

and so on; from which the required result follows immediately. This also shows that $p_1 = \mu$ is a factor of every $p_n(\mu)$ as is of course obvious otherwise.

(ii)

$$(23) \quad h_n(\lambda, \mu) - h_r(\lambda, \mu) = (h_{n-1} + h_{r-1})(h_{n-2} + h_{r-2}) \cdots (h_{n-r} + h_0)(h_{n-r} - h_0),$$

$$(23') \quad p_n(\mu) - p_r(\mu) = (p_{n-1} + p_{r-1})(p_{n-2} + p_{r-2}) \cdots (p_{n-r+1} + p_1)\, p_{n-r}^2.$$

For

$$\begin{aligned} h_s(\lambda, \mu) - h_t(\lambda, \mu) &= h_{s-1}^2(\lambda, \mu) - h_{t-1}^2(\lambda, \mu) \\ &= [h_{s-1}(\lambda, \mu) + h_{t-1}(\lambda, \mu)]\, [h_{s-1}(\lambda, \mu) - h_{t-1}(\lambda, \mu)]. \end{aligned}$$

An immediate consequence of (23) is that $h_n(\lambda, \mu)$, $(n = k, k+1, k+2, \ldots)$ all have the same value for values of λ and μ for which

$$h_r(\lambda, \mu) + h_{r-1}(\lambda, \mu) = 0, \qquad (r = 1, 2, \cdots k),$$

an important particular case of which is

$$p_n(-2) = 2, \qquad (n \gtreqless 2).$$

(iii) Differentiating (21) with regard to λ^2 and μ, we readily prove by induction that

$$(24) \quad \frac{\partial h_n}{\partial \mu} = 1 + 2h_{n-1} + 2^2 h_{n-1} h_{n-2} + 2^3 h_{n-1} h_{n-2} h_{n-3} + \cdots$$

$$\cdots + 2^{n-1} h_{n-1} h_{n-2} \cdots h_1,$$

$$(24') \qquad \frac{\partial h_n}{\partial \lambda^2} = 2^{n-1} h_{n-1} h_{n-2} \cdots h_1,$$

an interesting particular case of which is $p'_n(-2) = -(2^{2n-1} + 1)/3$, $(n \geqq 2)$.

(iv) *If*

$$(25) \qquad |\lambda^2 + \mu| \gtreqless \tfrac{1}{2} + \sqrt{|\mu| + \tfrac{1}{4}},$$

then

$$(26) \qquad |h_n(\lambda, \mu)| \gtreqless |\lambda^2 + \mu|^2 - |\mu| \gtreqless |\mu|^{\frac{1}{2}}, \qquad (n \gtreqless 2),$$

and, in particular, if $|\mu| \geqq 2$,

$$(26') \qquad |p_n(\mu)| \gtreqless |\mu|^2 - |\mu| \gtreqless |\mu|^{\frac{1}{2}}, \qquad (n \gtreqless 2).$$

If (25) is satisfied, then

$$|\mu| + |\mu|^{1/2} < |\mu| + \tfrac{1}{2} + \sqrt{|\mu| + \tfrac{1}{4}} = (\tfrac{1}{2} + \sqrt{|\mu| + \tfrac{1}{4}})^2 \gtreqless 1;$$

hence

$$|\lambda^2 + \mu|^2 \gtreqless |\mu| + |\mu|^{1/2},$$

or

$$|\lambda^2 + \mu|^2 - |\mu| \gtreqless |\mu|^{1/2} \gtreqless 0.$$

Now, for $n = 2$,

$$|h_2(\lambda, \mu)| = |\mu + (\lambda^2 + \mu)^2| \overline{\geqq} |\lambda^2 + \mu|^2 - |\mu| \overline{\geqq} |\mu|^{1/2}.$$

Let us assume, therefore, that (26) is true for 2, 3, ... n; then

$$\begin{aligned} |h_{n+1}(\lambda, \mu)| &= |h_n^2(\lambda, \mu) + \mu| \overline{\geqq} |h_n^2(\lambda, \mu)| - |\mu| \\ &\overline{\geqq} |\lambda^2 + \mu|^4 - 2|\mu|\,|\lambda^2 + \mu|^2 + |\mu|^2 - |\mu| \\ &= |\lambda^2 + \mu|^2 - |\mu| + [\,|\lambda^2 + \mu|^2 - (|\mu| + \tfrac{1}{2})]^2 - |\mu| - \tfrac{1}{4} \\ &\overline{\geqq} |\lambda^2 + \mu|^2 - |\mu| \overline{\geqq} |\mu|^{1/2} \end{aligned}$$

by (25). Equation (26) then follows by induction.

If $\lambda = 0$, (25) becomes

$$|\mu| \overline{\geqq} \tfrac{1}{2} + \sqrt{|\mu| + \tfrac{1}{4}},$$

or $|\mu|^2 \overline{\geqq} 2|\mu|$, i. e., $|\mu| \overline{\geqq} 2$; and (26) becomes (26′).

Since $|\lambda^2 + \mu| \overline{\geqq} \big|\,|\lambda|^2 - |\mu|\,\big|$, (25) is satisfied by

$$|\lambda|^2 \overline{\geqq} |\mu| + \tfrac{1}{2} + \sqrt{|\mu| + \tfrac{1}{4}}, \qquad |\lambda|^2 > |\mu|,$$

or

$$|\lambda|^2 \leqq |\mu| - \tfrac{1}{2} - \sqrt{|\mu| + \tfrac{1}{4}}, \qquad |\lambda|^2 < |\mu| \geqq 2.$$

(v) It follows immediately from (26′) that *the absolute value of every root of* $p_n(\mu)$, *except* $\mu = 0$, *is less than* 2 *and therefore corresponds to a value of* $|\alpha| < 1$ *and, if* α *is real, to a value of* α *between* -1 *and* 0.

(vi) *If* μ_n *is the real negative root of* p_n *of greatest absolute value, there is one, and only one, real root* μ_{n+1} *of* p_{n+1} *between* -2 *and* μ_n.

When $\mu = -2$, we have already seen that $p_n = 2 > 0$, and if $\mu = \mu_{n-1}$, then $p_n = \mu_{n-1} + p_{n-1}^2 = \mu_{n-1} < 0$; there is therefore at least one real root of p_n between -2 and μ_{n-1}.

If we differentiate (21′) twice, we get

$$p_n''(\mu) = 2[p_{n-1}(\mu)]^2 + 2p_{n-1}(\mu)\,p_{n-1}''(\mu),$$

so that, if μ is real, p''_n is positive if p_{n-1} and p''_{n-1} have the same sign. For $n=3$, $\mu_2=-1$; and, for $\mu<-1$, we evidently have $p''_3 \geqq 0$, so that to the left of $\mu=\mu_3$ both p_3 and p''_3 are positive. It follows that, to the left of μ_3, $p''_4 \geqq 0$; and so on. We can therefore conclude that p_n has one, and only one, real root between -2 and μ_{n-1} as otherwise p''_n, which is equal to or greater than 0, would change sign for some value of μ between these limits.

It is also easy to show that $\mu_n \to -2$ as $n \to \infty$. For

$$p_n(-2) = 2, \qquad p_n(\mu_{n-1}) = \mu_{n-1} < -1,$$

while $p''_n(\mu) \geqq 0$ between -2 and μ_{n-1} so that the graph of $p_n(\mu)$ is concave upwards between these limits; μ_n therefore lies to the left of the line joining $(-2, 2)$ to $(\mu_{n-1}, -1)$, whence

$$\mu_{n-1} - \mu_n > \tfrac{1}{3}(\mu_{n-1}+2).$$

(vii) *If μ and λ are real and positive, and if $\mu \leqq \frac{1}{4}$ and $\mu+\lambda^2-\lambda<0$ or $\lambda<\mu+\lambda^2<\frac{1}{2}$, then*

$$\lim_{n=\infty} h_n(\lambda, \mu) = \tfrac{1}{2} - \sqrt{\tfrac{1}{4}-\mu}.$$

For all other positive values of μ and λ, $h_n(\lambda,\mu) \to \infty$ as $n \to \infty$.

From (23) with $r = n-1$ we have

$$h_n - h_{n-1} = (h_{n-1}+h_{n-2})(h_{n-2}+h_{n-3}) \cdots (h_1+h_0)(\mu+\lambda^2-\lambda).$$

If $\mu+\lambda^2-\lambda<0$, which requires $\mu<1/4$, the h's therefore form a decreasing sequence of positive quantities and so approach a finite limit. If $\mu+\lambda^2-\lambda>0$, they form an increasing sequence. If the sequence of h's has a finite limit, then, since $h_n = \mu + h^2_{n-1}$, we must have $l = \mu + l^2$. Moreover, since $h_{n-1}+h_{n-2} \to 2l$, we see immediately from (23) that $l<1/2$, and hence $l = \frac{1}{2} - \sqrt{\frac{1}{4}-\mu}$. A finite limit can therefore only exist if $\mu<1/4$.

If $\mu<1/4$ and $h_1 \equiv \mu+\lambda^2+1/2$, then

$$h_2 = \mu + (\mu+\lambda^2)^2 < \tfrac{1}{2},$$

and therefore, by an easy induction, $h_n < 1/2$ for every n. On the other hand, if $\lambda < \mu+\lambda^2 > 1/2$, the h's form an increasing sequence which cannot have a finite limit since $h_{n-1}+h_{n-2}$ is greater than unity.

(viii) *If* $\mu > 1/4$, $\lambda > 0$ *and* $\dfrac{1}{x} = \lim\limits_{n=\infty} k_n(\lambda, \mu)$, *then* $\lambda = h(x, \mu)$.

In the first place, $k(\lambda, \mu) \equiv \lim\limits_{n=\infty} k_n(\lambda, \mu)$ exists and is finite. For, under the given conditions, h_n increases indefinitely with n and hence the condition of (19) and (20) of § 5 that $h_n > 1 + 2^{n-r}\mu^{\frac{1}{2}}$ is satisfied for n greater than some finite value of r. The value of the limit is moreover greater than unity; for we can write

$$h_n > (1+\varepsilon)^{2n-1}, \qquad (\varepsilon > 0)$$

for some n sufficiently large and ε sufficiently small, whence

$$h_{n+1} > \mu + (1+\varepsilon)^{2n} > (1+\varepsilon)^{2n},$$

so that the inequality also holds, with the same ε, for every subsequent value of n.

We shall now show that $\partial k_n/\partial\lambda$ approaches a finite value as $n \to \infty$. From (17) and (24′) we have

$$\frac{1}{2\lambda}\frac{\partial k_n}{\partial \lambda} = \frac{\partial k_n}{\partial \lambda^2} = k_n \frac{h_{n-1}h_{n-2}\cdots h_1}{h_n} \equiv k_n(\lambda)\, q_n(\lambda)$$

say. Now

$$\frac{q_{n+1}}{q_n} = \frac{h_n^2}{h_{n+1}} = \frac{1}{1+\dfrac{\mu}{h_n^2}},$$

and $\sum \mu/h_n^2$ converges since $h_n > 1 + 2^{n-r}\mu^{\frac{1}{2}}$ for n sufficiently large. Hence q_n approaches a definite finite limit as $n \to \infty$, and this limit is never zero. Moreover, k_n and q_n approach their limits uniformly as regards λ since in both cases the convergence was obtained by comparison with series which are independent of λ. Hence

$$\frac{\partial k(\lambda,\mu)}{\partial\lambda} = 2\lambda k(\lambda,\mu)\, q(\lambda,\mu).$$

It follows that, in any interval $0 < \varepsilon \leqq \lambda \leqq N$ in which ε is as small and N as great as we please, $x \equiv 1/k(\lambda, \mu)$ is a uniformly continuous function of λ and

possesses a derivative which is nowhere zero. There exists therefore a unique single-valued inverse function $\lambda = H(x, \mu)$. Now we saw in § 5 that $k(\lambda, \mu) = 1/x$ if λ is admissable as a value of $h(x, \mu)$; also $h(x, \mu) \to \infty$ as $x \to 0$; hence, if N is taken large enough, the range of values for $h(x, \mu)$ will overlap that for $H(x, \mu)$, and in the common part of their ranges these two functions have the same value. Both functions satisfy the same functional equation, namely $h(x^2) = \mu + h^2(x)$; hence, if a is so small that a and $a^{1/2}$ both lie in the common range, so will also $a^{1/4}$ provided always that $h(x, \mu)$ and $H(x, \mu)$ do not vanish; and so on. The two functions therefore have the same range of definition and this range is from ε to the point at which they vanish or, if they do not vanish, from ε to 1. We can then use (17′) to calculate the zero of $g(x)$ provided $\mu > 1/4$, and *the radius of convergence of $g(x)$ is less than unity for real values of $2\alpha \equiv \mu > 1/4$, and it is equal to unity for* $0 < \mu \leqq 1/4$.

7. **Solutions which have a singular point at the origin.** If $g(x)$ has a pole of order n at $x = 0$, then

$$G(x) = x^n g(x)$$

is regular there and $G(0) \neq 0$. The function $G(x)$ satisfies the functional equation

$$G(x^2) = 2\alpha x^{2n+1} + G^2(x), \tag{27}$$

so that $G(0) = 1$. Since $2n+1$ is integral whenever n is an integral multiple of 1/2, the discussion of this equation will also embrace those solutions which have an algebroid pole or zero whose order is of the form $m/2$.

In place of (27) we shall consider the more general equation

$$F(x^2) = 2(\alpha^0 + \alpha' x + \alpha'' x^2 + \cdots + \alpha^{(n)} x^n) + F^2(x), \tag{28}$$

and we shall only consider solutions which are regular at the $x = 0$. Substituting in this equation

$$F(x) = a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \cdots \tag{29}$$

we obtain

$$
\begin{aligned}
a_0^2 &= a_0 - 2\alpha^0 \\
2a_0 a_1 &= 2\alpha' \\
2a_0 a_2 &= a_1 a_1 + a_1 + 2\alpha'' \\
&\cdots\cdots\cdots\cdots \\
2a_0 a_n &= \sum_{r=1}^{n-1} a_r a_{n-r} + a_{n/2} + 2\alpha^n \\
2a_0 a_m &= \sum_{r=1}^{m-1} a_r a_{m-r} + a_{m/2} \qquad\qquad (m > n).
\end{aligned}
$$

If $a_0 = 0$, then $\alpha^0 = 0 = \alpha'$, so that $F_1(x) = F(x)/x$ is regular at $x = 0$ and satisfies the equation

$$F_1(x^2) = 2(\alpha'' + \alpha''' x + \cdots + \alpha^{(n)} x^{n-2}) + F_1^2(x);$$

we may therefore assume $a_0 \neq 0$ without real loss of generality.

The discussion of the convergence of (29) is very similar to that already given in § 3 and is therefore given here merely in outline. Let

$$a_r = a_0 b_r, \quad |b_r| = \beta_r, \quad |a_0| = \alpha_0, \quad |a^{(r)}| = \alpha_0^2 \beta^{(r)},$$

then

$$2\beta_m \leqq \sum \beta_r \beta_{m-r} + \frac{1}{\alpha_0}\beta_{m/2} + 2\beta^{(m)},$$

the last term being absent if $m > n$. There are two cases to consider.

(i) Suppose $1/\alpha_0 \leqq 1$. Let

$$\gamma_1 = \gamma, \qquad 2\gamma_m = \sum \gamma_r \gamma_{m-r} + \gamma_{m/2},$$

where γ is chosen so large that $\beta_m \leqq \gamma_m$ $(m = 1, 2, \ldots n)$. If this inequality holds for all values of the subscript from 1 to $m - 1 \geqq n$, then

$$2\beta_m \leqq \sum \gamma_m \gamma_{m-r} + \frac{1}{\alpha_0}\gamma_{m/2} \leqq \sum \gamma_r \gamma_{m-r} + \gamma_{m/2} = 2\gamma_m,$$

so that it also holds for every value of m. The proof that the series converges is then exactly as in § 3.

(ii) Suppose that $1/\alpha_0 > 1$. Then, γ being chosen as before, let

$$\gamma_1 = \gamma, \qquad 2\gamma_m = \sum \gamma_r \gamma_{m-r} + \frac{1}{\alpha_0}\gamma_{m/2},$$

so that

$$2\beta_m \leqq \sum \gamma_r \gamma_{m-r} + \frac{1}{\alpha_0}\gamma_{m/2} = 2\gamma_m, \qquad (m > n).$$

Let δ be a positive quantity satisfying the conditions

$$\delta \gtreqqless \alpha_0 \gamma, \qquad \delta \gtreqqless 1,$$

and set $\delta_1 = \delta$, $\delta_m = \sum \delta_r \delta_{m-r}$; then $\delta_m \gtreqqless \alpha_0^m \gamma_m$. This inequality is true when $m = 1$; suppose it is true for all values of the subscript up to $m-1$, then

$$2\gamma_m = \sum \gamma_r \gamma_{m-r} + \frac{1}{\alpha_0}\gamma_{m/2} \leqq \sum \left(\frac{1}{\alpha_0}\right)^m \delta_r \delta_{m-r} + \left(\frac{1}{\alpha_0}\right)^{\frac{m+2}{2}} \delta_{m/2}$$

$$= \left(\frac{1}{\alpha_0}\right)^m \delta_m + \left(\frac{1}{\alpha_0}\right)^{\frac{m+2}{2}} \delta_{m/2}.$$

But $\delta_{m/2} \leqq \delta_m$ and $m > (m+2)/2$; hence the inequality holds for every value of m. It then follows exactly as before that (29) converges for $|x| < \alpha_0/4\delta$.

It is not difficult to show* that no solution of (29) can be an entire function unless it is a polynomial. It then follows that the original functional equation (5) cannot have a solution of the form

$$g(x) = a_m x^m + \cdots + a_0 + \sum_1^\infty a_{-r} x^{-r}$$

in which the infinite series does not terminate. For, if there were such a solution, $G\left(\frac{1}{x}\right) = g(x)/x^m$ would be an entire function of $z = 1/x$ which

* One method of proof is to consider the manner in which $|F(x)|$ increases with $|x|$.

satisfies the equation

$$G(z^2) = 2\alpha z^{2m-1} + G^2(z)$$

and is not a polynomial.

8. **Conclusion.** The method of calculating the zeros of $g(x)$ which was given in § 5 may also be used to calculate the value of $g(x)$ for any real x and α. When this is carried out, it is immediately seen that comparatively few steps of the limiting process are required to give a fairly accurate result. If the number of steps required for a given degree of accuracy has been ascertained, the process may by profitably inverted, e. g., if four steps are sufficient, we may set

$$h(x) = [[[[x^{-8} - 2\alpha]^{\frac{1}{2}} - 2\alpha]^{\frac{1}{2}} - 2\alpha]^{\frac{1}{2}} - 2\alpha]^{\frac{1}{2}},$$

or in general

$$h(x) = [[\cdots[[x^{-2^{n-1}} - 2\alpha]^{\frac{1}{2}} - 2\alpha]^{\frac{1}{2}} \cdots - 2\alpha]^{\frac{1}{2}} - 2\alpha]^{\frac{1}{2}},$$

where the square root is extracted n times. If n is taken so large that $[x^{-2^{n-1}} - 2\alpha]^{\frac{1}{2}} = x^{-2^{n-2}}$ to the degree of accuracy required, then nothing is gained by increasing the number of steps beyond n.

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